HALF-INTEGER INDICES FOR ONE-DIMENSIONAL GAPLESS QUANTUM WALKS

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§1. Preliminaries

This talk is based mainly on index theory for ***chiral-symmetric unitary operators***. More precisely, we shall focus on an abstract unitary operator U on a Hilbert space $\mathcal{H}$, which satisfies the following algebraic condition;

$$U^* = \varGamma U \varGamma, \tag{1}$$

where $\varGamma$ can be any unitary self-adjoint operator on $\mathcal{H}$. Note that $\varGamma$ allows us to decompose the underlying Hilbert space $\mathcal{H}$ into an orthogonal sum of the form $\mathcal{H} = \ker(\varGamma - 1) \oplus \ker(\varGamma + 1)$ as is well-known, and that the ***chiral-symmetry condition*** (1) implies that the spectrum of U, denoted by $\sigma(U)$, is symmetric about the real axis.

If U is a chiral-symmetric unitary operator satisfying (1) and if $R = (U + U^*)/2$ denotes the real part of U, then it follows from (1) that the self-adjoint operator R can be written as a diagonal block-operator matrix of the form $R = R_1 \oplus R_2$ with respect to $\mathcal{H} = \ker(\varGamma - 1) \oplus \ker(\varGamma + 1)$. We obtain

$$\ker(U \mp 1) = \ker(R \mp 1) = \ker(R_1 \mp 1) \oplus \ker(R_2 \mp 1). \tag{2}$$

This motivates us to introduce $\mathrm{ind}_\pm(\varGamma, U) := \dim\ker(R_1 \mp 1) - \dim\ker(R_2 \mp 1)$. Note that this formal index is well-defined, if $\pm 1 \notin \sigma_{\mathrm{ess}}(U)$. We get $|\mathrm{ind}_\pm(\varGamma, U)| \leq \dim\ker(U \mp 1)$ by (2).

§2. A concrete example

We consider the following block-operator matrices on the Hilbert space $\ell^2(\mathbb{Z}, \mathbb{C}^2) = \ell^2(\mathbb{Z}, \mathbb{C}) \oplus \ell^2(\mathbb{Z}, \mathbb{C})$ of square-summable $\mathbb{C}^2$-valued sequences on $\mathbb{Z}$:

$$\varGamma := \begin{pmatrix} 1 & 0 \\ 0 & L^* \end{pmatrix} \begin{pmatrix} p & \sqrt{1-p^2} \\ \sqrt{1-p^2} & -p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & L \end{pmatrix}, \qquad \varGamma' := \begin{pmatrix} a & \sqrt{1-a^2} \\ \sqrt{1-a^2} & -a \end{pmatrix}. \tag{3}$$

where L is the left-shift operator on $\ell^2(\mathbb{Z}, \mathbb{C})$, and where $p = (p(x))_{x \in \mathbb{Z}}$ and $a = (a(x))_{x \in \mathbb{Z}}$ are two arbitrary sequences taking values in $[-1, 1]$. Here, p, a are viewed as the multiplication operators on $\ell^2(\mathbb{Z}, \mathbb{C})$ in the obvious way. Since $\varGamma, \varGamma'$ are unitary self-adjoint by construction, the unitary operator $U := \varGamma \varGamma'$ satisfies the chiral-symmetry condition (1). This unitary operator is often referred to as the time-evolution operator of the so-called ***(one-dimensional) split-step quantum walk***.

Theorem 1. *Let us assume the existence of the following limits for each* $\star = -\infty, +\infty$:

$$p(\star) := \lim_{x\to\star} p(x), \qquad a(\star) := \lim_{x\to\star} a(x). \tag{4}$$

Then $\pm 1 \notin \sigma_{\mathrm{ess}}(U)$ *if and only if* $p(\star) \mp a(\star) \neq 0$ *for each* $\star = -\infty, +\infty$. *In this case, we have*

$$\mathrm{ind}_{\pm}(\varGamma, U) = \frac{\mathrm{sign}\,(p(+\infty) \mp a(+\infty)) - \mathrm{sign}\,(p(-\infty) \mp a(-\infty))}{2}, \tag{5}$$

where sign *denotes the sign function, and where we set* $\mathrm{sign}\, 0 := 0$.

A sketch of proof. It follows from the *half-step decomposition method*[1] that $\pm 1 \notin \sigma_{\mathrm{ess}}(U)$ if and only if $X_\pm := (\mp\sqrt{(1+p)(1 \mp a)} + \sqrt{(1 - p(\cdot - 1))(1 \pm a)}L^*)/2$ is a Fredholm operator. In this case, $\mathrm{ind}_{\pm}(\varGamma, U)$ coincides with the Fredholm index of $X_\pm$. The remaining part of the claim follows from the fact that $X_\pm$ can be identified with the orthogonal sum of two Toeplitz operators constructed from (4), since we have the invariance of the Fredholm index and essential spectum with respect to compact perturbations. □

§3. A generalisation of the index formula

A bounded operator A on $\mathcal{H}$ is called ***trace-compatible***, if $A^*A - AA^*$ is of trace-class. For such A, we define

$$\mathrm{ind}_t(A) := \mathrm{Tr}\,(e^{-tA^*A} - e^{-tAA^*}), \qquad t \in \mathbb{R}.$$

If the limit $w(A) := \lim_{t\to\infty} \mathrm{ind}_t(A)$ exists, then we call $w(A)$ the ***Witten index*** of A (see [BGGSS87] for details).

Theorem 2. *Let U be the time-evolution operator of the split-step quantum walk, and let us assume the existence of limits of the form* (4). *We also impose the following additional assumption;*

$$\sum_{x=0}^{\infty} |\zeta(-x) - \zeta(-\infty)| + \sum_{x=1}^{\infty} |\zeta(x) - \zeta(+\infty)| < \infty, \qquad \zeta = p, a. \tag{6}$$

Then $X_\pm$ is trace-compatible, and the Witten index $w(X_\pm)$ is given by the right hand side of (5).

A derivation of this index formula requires some scattering theoretic tools such as the spectral shift function. Note that $\mathrm{ind}_{\pm}(\varGamma, U)$ in (5) takes values from $\{-1, 0, 1\}$, since we assume $p(\star) \mp a(\star) \neq 0$ for each $\star = -\infty, +\infty$ as in Theorem 1. On the other hand we have $w(X_\pm) \in \{-1, 0, 1\} \cup \{-1/2, 1/2\}$, since we remove this assumption in Theorem 2.

[BGGSS87] D. Bollé, F. Gesztesy, H. Grosse, W. Schweiger, B. Simon, Witten index, axial anomaly, and krein's spectral shift function in supersymmetric quantum mechanics, J. Math. Phys. **28**(7), 1512–1525 (1987).

[CGWW21] C. Cedzich, T. Geib, A. H. Werner, R. F. Werner, Chiral floquet systems and quantum walks at half-period, Ann. Henri Poincaré **22**(2), 375–413 (2021).

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[1]This highly non-trivial step is one of the main topics of [CGWW21].