Stationary scattering theory for C^2 long-range potentials

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1. Setting

In this talk, based on a joint work with Erik Skibsted (Aarhus University), we discuss the stationary scattering theory for the Schrödinger operator

$$H = \frac{1}{2}p^2 + V + q \quad \text{on } \mathcal{H} = L^2(\mathbb{R}^d) \text{ with } d \geq 2.$$

Assumption. Let $V \in C^2(\mathbb{R}^d;\mathbb{R})$, $q \in L^\infty(\mathbb{R}^d;\mathbb{R})$, and assume there exists $\sigma \in (0,1)$ and $C > 0$ such that for any $|\alpha| \leq 2$

$$|\partial^\alpha V| \leq C\langle x\rangle^{-\sigma-|\alpha|}, \quad |q(x)| \leq C\langle x\rangle^{-1-\sigma}.$$

Remark. Ikebe–Isozaki '82 and Gâtel–Yafaev '99 discussed the stationary theory for C^4 and C^3 potentials, respectively. Hörmander called $V + q$ a *2-admissible potential*, and constructed the time-dependent theory for it.

2. Eikonal equation

Theorem. *Let $\lambda_0 > 0$. Then there exist $R > 0$ and*

$$S = (2\lambda)^{1/2}|x|(1+s) \in C^2\big((\lambda_0,\infty) \times \{|x| > R\}\big)$$

such that:

1. *S solves*
$$\frac{1}{2}|\nabla_x S(\lambda,x)|^2 + V(x) = \lambda;$$
2. *$S(\lambda,\cdot)$ is the distance from $\{|x| = R\}$ w.r.t. $g = 2(\lambda - V)\,\mathrm{d}x^2$;*
3. *There exists $C > 0$ such that for any $k + |\alpha| \leq 2$*
$$\big|\partial_\lambda^k \partial_x^\alpha s(\lambda,x)\big| \leq C\lambda^{-1-k}\langle x\rangle^{-\sigma-|\alpha|}.$$

3. Stationary scattering matrix

Let $\mathcal{B}$, $\mathcal{B}^*$ and $\mathcal{B}_0^*$ be the Agmon–Hörmander spaces. Set for any $\xi \in \mathcal{G} := L^2(\mathbb{S}^{d-1})$

$$\phi_\pm^S[\xi](\lambda,x) = \frac{(2\pi)^{1/2}}{(2\lambda)^{1/4}}\chi(|x|/R)|x|^{-(d-1)/2}\mathrm{e}^{\pm\mathrm{i}S(\lambda,x)}\xi(x/|x|),$$

respectively, where $\chi \in C^\infty(\mathbb{R})$ with $\chi(t) = 0$ for $t \geq 1$ and $\chi(t) = 1$ for $t \geq 2$.

Theorem. *There exist continuous mappings $F^\pm\colon (\lambda_0,\infty) \times \mathcal{B} \to \mathcal{G}$ such that for any $(\lambda,\psi) \in (\lambda_0,\infty) \times \mathcal{B}$*

$$R(\lambda \pm \mathrm{i}0)\psi - \phi_\pm^S\big[F^\pm(\lambda)\psi\big](\lambda,\cdot) \in \mathcal{B}_0^*,$$

respectively. Moreover, they satisfy for any $\lambda > \lambda_0$:

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1. *One has*
$$(H-\lambda)F^{\pm}(\lambda)^*=0,\quad F^{\pm}(\lambda)^*F^{\pm}(\lambda)=\delta(H-\lambda),$$
respectively, where $\delta(H-\lambda)=\pi^{-1}\operatorname{Im}R(\lambda+\mathrm{i}0)\in\mathcal{L}(\mathcal{B},\mathcal{B}^*)$.

2. *For any* $\lambda>\lambda_0$, $F^{\pm}(\lambda)\colon\mathcal{B}\to\mathcal{G}$ *are surjective.*

Definition. 1. $F^{\pm}(\lambda)\colon\mathcal{B}\to\mathcal{G}$ are the *stationary wave operators*, and they are *complete* if they are surjective.

2. $F^{\pm}(\lambda)^*\colon\mathcal{G}\to\mathcal{B}^*$ are the *stationary wave matrices.*
3. The *stationary scattering matrix* $\mathsf{S}(\lambda)$ is a unitary operator on $\mathcal{G}$ obeying
$$F^{+}(\lambda)=\mathsf{S}(\lambda)F^{-}(\lambda).$$

Corollary. *The stationary scattering matrix* $\mathsf{S}(\lambda)$ *uniquely exists. Moreover, it is strongly continuous in* λ.

4. Asymptotics of generalized eigenfuctions

Set
$$\mathcal{E}_\lambda=\bigl\{\phi\in\mathcal{B}^*;\ (H-\lambda)\phi=0\text{ in the distributional sense}\bigr\}.$$

Theorem. *1. For any one component of* $(\xi_-,\xi_+,\phi)\in\mathcal{G}\times\mathcal{G}\times\mathcal{E}_\lambda$ *the other two components uniquely exist such that*
$$\phi-\phi^S_+[\xi_+](\lambda,\cdot)+\phi^S_-[\xi_-](\lambda,\cdot)\in\mathcal{B}^*_0.$$

2. *The above correspondence is given by the formulas*
$$\phi=2\pi\mathrm{i}F^{\pm}(\lambda)^*\xi_{\pm},\quad \xi_+=\mathsf{S}(\lambda)\xi_-.$$

3. $F^{\pm}(\lambda)^*\colon\mathcal{G}\to\mathcal{E}_\lambda\subseteq\mathcal{B}^*$ *are topological linear isomorphisms.*

4. $\delta(H-\lambda)\colon\mathcal{B}\to\mathcal{E}_\lambda$ *is surjective.*

5. Generalized Fourier transforms

Set $I=(\lambda_0,\infty)$, and we let
$$\mathcal{H}_I=P_H(I)\mathcal{H},\quad H_I=H_{|\mathcal{H}_I},\quad \widetilde{\mathcal{H}}_I=L^2(I,\mathrm{d}\lambda;\mathcal{G}).$$
Thanks to the continuity of $F^{\pm}$, we have
$$\mathcal{F}^{\pm}_0:=\int_I^{\oplus}F^{\pm}(\lambda)\,\mathrm{d}\lambda\colon\mathcal{B}\to C(I;\mathcal{G}).$$

Theorem. *The above* $\mathcal{F}^{\pm}_0$ *induce unitary operators* $\mathcal{F}^{\pm}\colon\mathcal{H}_I\to\widetilde{\mathcal{H}}_I$*, respectively. Moreover, they satisfy*
$$\mathcal{F}^{\pm}H_I(\mathcal{F}^{\pm})^*=M_\lambda,$$
respectively.

We will also discuss the time-dependent scattering theory, and present stationary representation of the time-dependent wave operators. Key ingredients of the proofs are (1) Hörmander's regularization of V, (2) Estimates for a solution to the eikonal equation, cf. Cruz–Skibsted '13, (3) The strong radiation bounds, and (4) The WKB approximation for $R(\lambda\pm\mathrm{i}0)$. We will mainly focus on (3) in the rest of the talk.